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LITERATURE CITED

1. V. S. Voitsenya, "Translational motion of a body over an interface between two fluids," Izv. Vyssh. Uchebn. Zaved., Mat., No. 2 (1963).
2. I. V. Sturova, "Wave motions in a fluid with discrete stratification during flow around an immersed body," in: Numerical Methods of Continuum Mechanics [in Russian], Vol. 6 (No. 3), Izd. VTs Sib. Otd. Akad. Nauk SSSR, Novosibirsk (1975).
3. I. V. Sturova and V. A. Sukharev, "The plane problem of wave motions in a continuously stratified fluid during flow around an immersed body," Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza, No. 4 (1978).
4. J. V. Droughton and C. F. Chen, "The channel flow of a density-stratified fluid about immersed bodies," Trans. Am. Soc. Mech. Eng., Ser. D: J. Basic Eng., 94, No. 1 (1972).
5. T. I. Gural', O. A. Ershov, A. G. Sorits, and B. A. Fomenko, "Holographic interferometric study of temperature and concentration fields," in: Heat and Mass Transfer [in Russian], Vol. 10, Naukova Dumka, Kiev (1976).
6. V. N. Nekrasov, A. M. Trokhan, and Yu. D. Chashechkin, "Generation of internal waves in a plane-layered medium by a uniformly moving hydrodynamic source (three-dimensional problem)," in: Wave Diffraction and Propagation Theory [in Russian], Vol. 3, Moscow (1977).
7. A. E. Kochin, I. A. Kibel', and N. V. Roze, Theoretical Fluid Mechanics [in Russian], Part 1, Fizmatgiz, Moscow (1963).
8. J. S. Turner, Buoyancy Effects in Fluids, Cambridge Univ. Press (1973).

ONE FORM OF THE EQUATIONS OF HYDRODYNAMICS OF AN IDEAL INCOMPRESSIBLE FLUID AND THE VARIATIONAL PRINCIPLE FOR NONSTEADY FLOW WITH A FREE SURFACE

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In the investigation of nonsteady flows having a free surface there are well-known difficulties [1] connected with the formulation of the problems in the traditional statements of Euler or Lagrange.

Using the "Clebsch potentials" χ, μ, and λ one can write the equations for an ideal incompressible fluid in the form [2, 3]

$$\partial v_i/\partial x_i = 0; \quad (1)$$

$$\partial\mu/\partial t + v_i\partial\mu/\partial x_i = 0; \quad (2)$$

$$\partial\lambda/\partial t + v_i\partial\lambda/\partial x_i = 0, \quad (3)$$

where the velocity components v_i are expressed by the equations

$$v_i = \partial\chi/\partial x_i + \lambda\partial\mu/\partial x_i \ (i = 1, 2, 3). \quad (4)$$

Here and later in writing the equations we use the rule of summation over double repeated ("dummy") indices.

For the pressure p there is the expression

$$p = -\rho\left(\frac{\partial\chi}{\partial t} + \lambda\frac{\partial\mu}{\partial t} + \frac{1}{2}v_i^2\right) \ (i = 1, 2, 3), \quad (5)$$

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where ρ is the fluid density. Here the surfaces of λ = const and μ = const are vortex surfaces.

We change to new independent variables x_1, x_2, μ, taking χ, λ, and x_3 as the unknowns. After the corresponding transformations, from (4) we obtain the following expressions for the velocity components:

$$v_i = \partial\chi/\partial x_i - \alpha_i(\partial\chi/\partial\mu + \lambda) \;\; (i = 1,2), \;\; v_3 = \alpha_3(\partial\chi/\partial\mu + \lambda), \tag{6}$$

where $\alpha_i = \left(\frac{\partial x_3}{\partial x_i}\right)\Big/\left(\frac{\partial x_3}{\partial \mu}\right)$ $(i=1,2)$; $\alpha_3 = 1\Big/\left(\frac{\partial x_3}{\partial\mu}\right)$. In place of χ, λ we introduce the new functions γ, η:

$$\chi = \gamma + \eta, \;\; \lambda = -\partial\eta/\partial\mu. \tag{7}$$

Then from (6) we obtain

$$v_i = \frac{\partial}{\partial x_i}(\gamma + \eta) - \alpha_i\frac{\partial\gamma}{\partial\mu} \;\; (i=1,2), \quad v_3 = \alpha_3\frac{\partial\gamma}{\partial\mu}. \tag{8}$$

Equations (1)-(3) and (5) in the new variables, with allowance for (7) and (8), take the respective forms

$$\frac{\partial v_i}{\partial x_i} - \alpha_i\frac{\partial v_i}{\partial\mu} + \alpha_3\frac{\partial v_3}{\partial\mu} = 0 \quad (i=1,2); \tag{9}$$

$$\frac{\partial x_3}{\partial t} + v_i\frac{\partial x_3}{\partial x_i} = v_3 \quad (i=1,2); \tag{10}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial\eta}{\partial\mu}\right) + v_i\frac{\partial}{\partial x_i}\left(\frac{\partial\eta}{\partial\mu}\right) = 0 \quad (i=1,2); \tag{11}$$

$$p = -\rho\left[\frac{\partial}{\partial t}(\gamma+\eta) - v_3\frac{\partial x_3}{\partial t} + \frac{1}{2}v_i^2\right] \quad (i=1,2,3). \tag{12}$$

Equation (10) (the kinematic condition) requires that fluid particles which initially lay at the vortex surface μ = const remain at it during the entire time of motion.

Equations (9)-(11), in which v_i are determined by Eqs. (8), represent a system for the determination of γ, x_3, η. By combining Eqs. (9)-(11) we can obtain a system of solvable equations of divergent form, which proves useful in the numerical solution of problems [4, 5]. Multiplying Eq. (9) by $\partial x_3/\partial\mu$, after substitution of the values of α_i of (6) we obtain

$$\frac{\partial}{\partial\mu}\left(v_3 - v_i\frac{\partial x_3}{\partial x_i}\right) + \frac{\partial}{\partial x_i}\left(v_i\frac{\partial x_3}{\partial\mu}\right) = 0 \quad (i=1,2). \tag{13}$$

Substituting (10) into (13), we find

$$\frac{\partial}{\partial t}\left(\frac{\partial x_3}{\partial\mu}\right) + \frac{\partial}{\partial x_i}\left(v_i\frac{\partial x_3}{\partial\mu}\right) = 0 \quad (i=1,2). \tag{14}$$

Using (8) and (13), Eq. (11) can be reduced to the form

$$\frac{\partial}{\partial\mu}\left(\frac{\partial\gamma}{\partial t} + \frac{p}{\rho}\right) + \frac{\partial}{\partial x_i}\left(v_3 v_i\frac{\partial x_3}{\partial\mu}\right) = 0 \quad (i=1,2), \tag{15}$$

where p is determined, with allowance for (10), by the expression

$$p = -\rho\left[\frac{\partial}{\partial t}(\gamma+\eta) + \frac{1}{2}(v_i^2 - v_3^2) + v_3 v_i\frac{\partial x_3}{\partial x_i}\right] \quad (i=1,2). \tag{16}$$

It is simple to verify the equivalence of Eqs. (11) and (15). If we eliminate p from (15) using the expression (16) and separate out of the resulting equation the term

$$v_3\left[\frac{\partial}{\partial\mu}\left(v_3 - v_i\frac{\partial x_3}{\partial x_i}\right) + \frac{\partial}{\partial x_i}\left(v_i\frac{\partial x_3}{\partial\mu}\right)\right],$$

which is reduced to zero by virtue of (13), then after substituting the expressions (8) into the remaining part of the equation we obtain (11). Thus, in place of (9)-(11) we have the system of solvable equations (13)-(15).

The proposed form of writing is convenient in the analysis of flows having a free surface, both potential and vortical, bounded by vortex surfaces with $\mu = \mu_1$ = const and $\mu = \mu_2$ = const. The introduction of outside forces having a potential offers no difficulty. The advantage of the given formulation consists in the fact that the solution of the system (13)-(15) is sought in a fixed region of variation of the variables x_1, x_2, μ. And the region of flow is defined physically by Eq. (14). The original system (1)-(3) does not contain this equation in explicit form.

It should be noted that the order of the representation (6) is increased with the help of the substitution (7). Since $\lambda = \partial\eta/\partial\mu$, for given λ and χ the functions γ and η can be determined with the accuracy of an arbitrary function $c_1(x_1, x_2, t)$. Consequently, the arbitrarity in the determination of γ and η has no importance for the unique solution of the problem. Therefore, one of these functions can be assigned arbitrarily at either boundary ($\mu = \mu_1$ or $\mu = \mu_2$), for example, $\gamma = 0$.

For the case of the flow of a fluid with a free surface over a stationary bottom the boundary conditions at the free surface ($p = 0$ at $\mu = \mu_2$) and at the bottom ($x_3 = f(x_1, x_2)$ at $\mu = \mu_1$) can be written in the adopted variables in the form

$$x_3 = f(x_1,\ x_2); \tag{17}$$

$$v_i\partial x_3/\partial x_i - v_3 = 0\ \ (i = 1,2)\quad \text{at}\quad \mu = \mu_1; \tag{18}$$

$$\gamma = 0; \tag{19}$$

$$\frac{\partial\eta}{\partial t} + \frac{1}{2}\left(v_i^2 - v_3^2\right) + v_3 v_i\frac{\partial x_3}{\partial x_i} = 0\quad (i = 1, 2)\quad \text{at}\quad \mu = \mu_2. \tag{20}$$

In writing the condition $p = 0$ of (20) we allowed for the condition (19).

We note that the system (13)-(15) is not formally equivalent to the system (9)-(11). In fact, changing from Eqs. (13) and (14) back to (9) and (10), in place of (10) we obtain the condition

$$\frac{\partial}{\partial\mu}\left(\frac{\partial x_3}{\partial t} + v_i\frac{\partial x_3}{\partial x_i} - v_3\right) = 0,$$

from which we get

$$\frac{\partial x_3}{\partial t} + v_i\frac{\partial x_3}{\partial x_i} - v_3 = c_2(x_1, x_2, t).$$

Thus, equivalence of the systems requires that $c_2 \equiv 0$. In the integration of the system (13)-(15) this requirement is automatically satisfied in the assignment of the relation (10) at one of the boundaries μ = const. In the case of the boundary conditions considered above this relation acquires the form of (18).

We point out that the system (13)-(15) can be obtained directly from Lagrange's equations [2, 3] by replacing the two Lagrangian variables by the Eulerian variables x_1, x_2 with subsequent use of the substitution (8). In this case it turns out that the remaining Lagrangian variable coincides in meaning with the variable μ present in our equations.

This system can also be obtained from the variational principle given in [2]. Transformed to the variables x_1, x_2, μ, it takes the form

$$\delta M = 0,$$

where

$$M = \int_t \int_{x_1} \int_{x_2} \int_{\mu_1}^{\mu_2} L \frac{\partial x_3}{\partial \mu} d\mu dx_1 dx_2 dt; \tag{21}$$

$$L = \frac{\partial}{\partial t}(\gamma + \eta) - v_3 \frac{\partial x_3}{\partial t} + \frac{1}{2} v_i^2 \quad (i = 1, 2, 3);$$

v_i are determined by Eqs. (8). Varying the functional (21) with respect to γ, η, and x_3, we obtain Eqs. (13), (14), and (15), respectively. In this case the natural boundary conditions at the boundary surfaces $\mu = \mu_1$ and $\mu = \mu_2$ are determined as

$$\left[\frac{\partial}{\partial t}(\gamma + \eta) + \frac{1}{2}(v_i^2 - v_3^2) + v_3 v_i \frac{\partial x_3}{\partial x_i}\right] \delta x_3 = 0 \quad (i = 1, 2),$$

$$\left(\frac{\partial x_3}{\partial t} + v_i \frac{\partial x_3}{\partial x_i} - v_3\right) \delta\gamma = 0 \quad (i = 1, 2).$$

As is seen, the conditions (17)-(20) are a particular case of these conditions.

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LITERATURE CITED

1. R. D. Richtmyer and K. W. Morton, Difference Methods for Initial-Value Problems, Wiley, New York (1967).
2. R. L. Seliger and G. B. Whitham, "Variational principles in continuum mechanics," Proc. R. Soc., Ser. A, 305, 1-25 (1968).
3. J. Serrin, "Mathematical principles of classical fluid mechanis," in: Handbook of Physics, Vol. 8/1 (1959).
4. G. I. Marchuk, Numerical Methods in Weather Forecasting [in Russian], Gidrometeoizdat Leningrad (1967).
5. N. N. Yanenko, The Method of Fractional Steps for Solving Multidimensional Problems of Mathematical Physics [in Russian], Nauka, Novosibirsk (1967).